Table 2.1 The art of mathematical modeling: simplifying or refining the model as required

Model simplification	Model refinement
1. Restrict problem identification.	1. Expand the problem.
2. Neglect variables.	2. Consider additional variables.
3. Conglomerate effects of several variables.	3. Consider each variable in detail.
4. Set some variables to be constant.	4. Allow variation in the variables.
5. Assume simple (linear) relationships.	5. Consider nonlinear relationships.
6. Incorporate more assumptions.	6. Reduce the number of assumptions.

2.1 PROBLEMS

In Problems 1–8, the scenarios are vaguely stated. From these vague scenarios, identify a problem you would like to study. Which variables affect the behavior you have identified in the problem identification? Which variables are the most important? Remember, there are really no right answers.

1. The population growth of a single species.

2. A retail store intends to construct a new parking lot. How should the lot be illuminated?

3. A farmer wants to maximize the yield of a certain crop of food grown on his land. Has the farmer identified the correct problem? Discuss alternative objectives.

4. How would you design a lecture hall for a large class?

5. An object is to be dropped from a great height. When and how hard will it hit the ground?

6. How should a manufacturer of some product decide how many units of that product should be manufactured each year and how much to charge for each unit?

7. The United States Food and Drug Administration is interested in knowing if a new drug is effective in the control of a certain disease in the population.

8. How fast can a skier ski down a mountain slope?

For the scenarios presented in Problems 9–17, identify a problem worth studying and list the variables that affect the behavior you have identified. Which variables would be neglected completely? Which might be considered as constants initially? Can you identify any submodels you would want to study in detail? Identify any data you would want collected.

9. A botanist is interested in studying the shapes of leaves and the forces that mold them. She clips some leaves from the bottom of a white oak tree and finds the leaves to be rather broad and not very deeply indented. When she goes to the top of the tree, she finds very deeply indented leaves with hardly any broad expanse of blade.

10. Animals of different sizes work differently. Small ones have squeaky voices, their hearts beat faster, and they breathe more often than larger ones. On the other hand, the skeleton of a larger animal is more robustly built than that of a small animal. The ratio of the diameter to the length is greater in a larger animal than it is in a smaller one. Thus there are regular distortions in the proportions of animals as the size increases from small to large.

11. A physicist is interested in studying properties of light. He wants to understand the path of a ray of light as it travels through the air into a smooth lake, particularly at the interface of the two different media.

12. A company with a fleet of trucks faces increasing maintenance costs as the age and mileage of the trucks increase.

13. People are fixated by speed. Which computer systems offer the most speed?

14. How can we improve our ability to sign up for the best classes each term?

15. How should we save a portion of our earnings?

16. Consider a new company that is just getting started in producing a single product in a competitive market situation. Discuss some of the short-term and long-term goals the company might have as it enters into business. How do these goals affect employee job assignments? Would the company necessarily decide to maximize profits in the short run?

17. Discuss the differences between using a model to predict versus using one to explain a real-world system. Think of some situations in which you would like to explain a system. Likewise, imagine other situations in which you would want to predict a system.

2.1 PROJECTS

1. Consider the taste of brewed coffee. What are some of the variables affecting taste? Which variables might be neglected initially? Suppose you hold all variables fixed except water temperature. Most coffeepots use boiled water in some manner to extract the flavor from the ground coffee. Do you think boiled water is optimal for producing the best flavor? How would you test this submodel? What data would you collect and how would you gather them?

2. A transportation company is considering transporting people between skyscrapers in New York City via helicopter. You are hired as a consultant to determine the number of helicopters needed. Identify an appropriate problem precisely. Use the model-building process to identify the data you would like to have to determine the relationships between the variables you select. You may want to redefine your problem as you proceed.

3. Consider wine making. Suggest some objectives a commercial producer might have. Consider taste as a submodel. What are some of the variables affecting taste? Which variables might be neglected initially? How would you relate the remaining variables? What data would be useful to determine the relationships?

4. Should a couple buy or rent a home? As the cost of a mortgage rises, intuitively, it would seem that there is a point where it no longer pays to buy a house. What variables determine the total cost of a mortgage?

5. Consider the operation of a medical office. Records have to be kept on individual patients, and accounting procedures are a daily task. Should the office buy or lease a small computer system? Suggest objectives that might be considered. What variables would you consider? How would you relate the variables? What data would you like to have to determine the relationships between the variables you select? Why might solutions to this problem differ from office to office?

6. When should a person replace his or her vehicle? What factors should affect the decision? Which variables might be neglected initially? Identify the data you would like to have to determine the relationships among the variables you select.

7. How far can a person long jump? In the 1968 Olympic Games in Mexico City, Bob Beamon of the United States increased the record by a remarkable 10%, a record that stood through the 1996 Olympics. List the variables that affect the length of the jump. Do you think the low air density of Mexico City accounts for the 10% difference?

8. Is college a financially sound investment? Income is forfeited for 4 years, and the cost of college is extremely high. What factors determine the total cost of a college education? How would you determine the circumstances necessary for the investment to be profitable?

2.2 Modeling Using Proportionality

We introduced the concept of proportionality in Chapter 1 to model change. Recall that

$$y \propto x \text{ if and only if } y = kx \text{ for some constant } k \neq 0 \tag{2.1}$$

Of course, if $y \propto x$, then $x \propto y$ because the constant k in Equation (2.1) is not equal to zero and then $x = (\frac{1}{k})y$. The following are other examples of proportionality relationships:

$$y \propto x^2 \text{ if and only if } y = k_1 x^2 \text{ for } k_1 \text{ a constant} \tag{2.2}$$
$$y \propto \ln x \text{ if and only if } y = k_2 \ln x \text{ for } k_2 \text{ a constant} \tag{2.3}$$
$$y \propto e^x \text{ if and only if } y = k_3 e^x \text{ for } k_3 \text{ a constant} \tag{2.4}$$

In Equation (2.2), $y = kx^2, k \neq 0$, so we also have $x \propto y^{1/2}$ because $x = (\frac{1}{\sqrt{k}})y^{1/2}$. This leads us to consider how to link proportionalities together, a transitive rule for proportionality:

$$y \propto x \quad \text{and} \quad x \propto z, \text{ then} \quad y \propto z$$

Thus, any variables proportional to the same variables are proportional to one another.

Now let's explore a geometric interpretation of proportionality. In Equation (2.1), $y = kx$ yields $k = y/x$. Thus, k may be interpreted as the tangent of the angle θ depicted in Figure 2.8, and the relation $y \propto x$ defines a set of points along a line in the plane with angle of inclination θ.

Comparing the general form of a proportionality relationship $y = kx$ with the equation for a straight line $y = mx+b$, we can see that the graph of a proportionality relationship is a