into Eq. (2.7)). Hence, we have lost only 0.43 percent of the potential maximum profit by applying the results of our model, even though our actual production levels were quite a way off from the optimum values. Our model appears to be extremely robust in this regard. Furthermore, a similar conclusion should hold for many similar problems, since it is basically due to the fact that $\nabla f = 0$ at a critical point.

All of the previous sensitivity analysis calculations could also be performed using a computer algebra system. In fact, this is the preferred method, assuming that one is available. Figure 2.8 illustrates how the computer algebra system Maple can be used to compute the sensitivity $S(x_1, a)$. The calculations of the other sensitivities are similar.

Sensitivity analysis for the other elasticities could be carried out in the same manner. While the particulars will differ, the form of the function f suggests that each affects y in essentially the same manner. In particular, we have a high degree of confidence that our model will lead to a good (nearly optimal) decision about production levels even in the presence of small errors in the estimation of price elasticities.

We will say just a few words on the more general subject of robustness. Our model is based on a linear price structure. Certainly, this is only an approximation. However, in practical applications we are likely to proceed as follows. We begin with an educated guess about the size of the market for our new products and with a reasonable average sale price. Then we estimate elasticities either on the basis of past experience with similar situations or on the basis of limited marketing studies. We should be able to get reasonable estimates for these elasticities over a certain range of sales levels. This range presumably includes the optimal levels. So in effect we are simply making a linear approximation of a nonlinear function over a fairly small region. This sort of approximation is well known to exhibit robustness. After all, this is the whole idea behind calculus.

2.2 Lagrange Multipliers

In this section we begin to consider optimization problems with a more sophisticated structure. As we noted at the beginning of the previous section, complications arise in the solution of multivariable optimization models when the set over which we optimize becomes more complex. In real problems we are led to consider these more complicated models by the existence of constraints on the independent variables.

Example 2.2. We reconsider the color TV problem (Example 2.1) introduced in the previous section. There we assumed that the company has the potential to produce any number of TV sets per year. Now we will introduce constraints based on the available production capacity. Consideration of these two new products came about because the company plans to discontinue manufacture of some older models, thus providing excess capacity at its assembly plant. This excess capacity could be used to increase production of other existing product lines, but the company feels that the new products will be more profitable. It

Variables: s = number of 19–inch sets sold (per year)
t = number of 21–inch sets sold (per year)
p = selling price for a 19–inch set (\$)
q = selling price for a 21–inch set (\$)
C = cost of manufacturing sets (\$/year)
R = revenue from the sale of sets (\$/year)
P = profit from the sale of sets (\$/year)

Assumptions: $p = 339 - 0.01s - 0.003t$
$q = 399 - 0.004s - 0.01t$
$R = ps + qt$
$C = 400,000 + 195s + 225t$
$P = R - C$
$s \leq 5000$
$t \leq 8000$
$s + t \leq 10,000$
$s \geq 0$
$t \geq 0$

Objective: Maximize P

Figure 2.9: Results of step 1 for the color TV problem with constraints.

is estimated that the available production capacity will be sufficient to produce 10,000 sets per year ($\approx$ 200 per week). The company has an ample supply of 19–inch and 21–inch LCD panels and other standard components; however, the circuit boards necessary for constructing the sets are currently in short supply. Also, the 19–inch TV requires a different board than the 21–inch model because of the internal configuration, which cannot be changed without a major redesign, which the company is not prepared to undertake at this time. The supplier is able to supply 8,000 boards per year for the 21–inch model and 5,000 boards per year for the 19–inch model. Taking this information into account, how should the company set production levels?

Once again we will employ the five-step method. The results of step 1 are shown in Figure 2.9. The only change is the addition of several constraints on the decision variables s and t. Step 2 is to select the modeling approach.

This problem will be modeled as a multivariable constrained optimization problem and solved using the method of Lagrange multipliers.

We are given a function $y = f(x_1, \ldots, x_n)$ and a set of constraints. For the moment we will assume that these constraints can be expressed in the form of k functional equations

$$\begin{aligned} g_1(x_1, \ldots, x_n) &= c_1 \\ g_2(x_1, \ldots, x_n) &= c_2 \\ &\vdots \\ g_k(x_1, \ldots, x_n) &= c_k. \end{aligned}$$

Later on we will explain how to handle inequality constraints. Our job is to optimize

$$y = f(x_1, \ldots, x_n)$$

over the set

$$S = \{(x_1, \ldots, x_n) : \ g_i(x_1, \ldots, x_n) = c_i \text{ for all } i = 1, \ldots, k\}.$$

There is a theorem that states that at an extreme point $x \in S$, we must have

$$\nabla f = \lambda_1 \nabla g_1 + \cdots + \lambda_k \nabla g_k.$$

We call $\lambda_1, \ldots, \lambda_k$ the *Lagrange multipliers*. This theorem assumes that $\nabla g_1, \ldots, \nabla g_k$ are linearly independent vectors (see Edwards (1973), p. 113). Then in order to locate the max–min points of f on the set S, we must solve the n Lagrange multiplier equations

$$\begin{aligned} \frac{\partial f}{\partial x_1} &= \lambda_1 \frac{\partial g_1}{\partial x_1} + \cdots + \lambda_k \frac{\partial g_k}{\partial x_1} \\ &\vdots \\ \frac{\partial f}{\partial x_n} &= \lambda_1 \frac{\partial g_1}{\partial x_n} + \cdots + \lambda_k \frac{\partial g_k}{\partial x_n} \end{aligned}$$

together with the k constraint equations

$$\begin{aligned} g_1(x_1, \ldots, x_n) &= c_1 \\ &\vdots \\ g_k(x_1, \ldots, x_n) &= c_k \end{aligned}$$

for the variables $x_1, \ldots, x_n$ and $\lambda_1, \ldots, \lambda_k$. We must also check any exceptional points at which the gradient vectors $\nabla g_1, \ldots, \nabla g_k$ are not linearly independent.

The method of Lagrange multipliers is based on a geometrical interpretation of the gradient vector. Suppose for the moment that there is only one constraint equation,

$$g(x_1, \ldots, x_n) = c,$$

so that the Lagrange multiplier equation becomes

$$\nabla f = \lambda \nabla g.$$

The set $g = c$ is a curved surface of dimension $n-1$ in $\mathbb{R}^n$, and for any point $x \in S$ the gradient vector $\nabla g(x)$ is perpendicular to S at that point. The gradient vector ∇f always points in the direction in which f increases the fastest. At a local max or min, the direction in which f increases fastest must also be perpendicular to S, so at that point we must have ∇f and ∇g pointing along the same line; i.e., $\nabla f = \lambda \nabla g$.

In the case of several constraints, the geometrical argument is similar. Now the set S represents the intersection of the k level surfaces $g_1 = c_1, \ldots, g_k = c_k$. Each one of these is an $(n-1)$–dimensional subset of $\mathbb{R}^n$, so their intersection is an $(n-k)$–dimensional subset. At an extreme point, ∇f must be perpendicular to the set S. Therefore it must lie in the space spanned by the k vectors $\nabla g_1, \ldots, \nabla g_k$. The technical condition of linear independence ensures that the k vectors $\nabla g_1, \ldots, \nabla g_k$ actually do span a k–dimensional space. (In the case of a single constraint, linear independence simply means that $\nabla g \neq 0$.)

Example 2.3. Maximize $x + 2y + 3z$ over the set $x^2 + y^2 + z^2 = 3$.

This is a constrained multivariable optimization problem. Let

$$f(x, y, z) = x + 2y + 3z$$

denote the objective function, and let

$$g(x, y, z) = x^2 + y^2 + z^2$$

denote the constraint function. Compute

$$\begin{aligned} \nabla f &= (1, 2, 3) \\ \nabla g &= (2x, 2y, 2z). \end{aligned}$$

At the maximum, $\nabla f = \lambda \nabla g$; in other words,

$$\begin{aligned} 1 &= 2x\lambda \\ 2 &= 2y\lambda \\ 3 &= 2z\lambda. \end{aligned}$$

This gives three equations in four unknowns. Solving in terms of λ, we obtain

$$\begin{aligned} x &= 1/2\lambda \\ y &= 1/\lambda \\ z &= 3/2\lambda. \end{aligned}$$

Using the fact that

$$x^2 + y^2 + z^2 = 3,$$

we obtain a quadratic equation in λ, with two real roots. The root $\lambda = \sqrt{42}/6$ leads to

$$\begin{aligned} x &= \frac{1}{2\lambda} = \frac{\sqrt{42}}{14} \\ y &= \frac{1}{\lambda} = \frac{\sqrt{42}}{7} \\ z &= \frac{3}{2\lambda} = \frac{3\sqrt{42}}{14}, \end{aligned}$$

so that the point

$$a = \left(\frac{\sqrt{42}}{14}, \frac{\sqrt{42}}{7}, \frac{3\sqrt{42}}{14} \right)$$

is one candidate for the maximum. The other root, $\lambda = -\sqrt{42}/6$, leads to another candidate, $b = -a$. Since $\nabla g \neq 0$ everywhere on the constraint set $g = 3$, a and b are the only two candidates for the maximum. Since f is a continuous function on the closed and bounded set $g = 3$, f must attain its maximum and minimum on this set. Then, since

$$f(a) = \sqrt{42}, \qquad \text{and} \qquad f(b) = -\sqrt{42},$$

the point a is the maximum and b is the minimum. Consider the geometry of this example. The constraint set S defined by the equation

$$x^2 + y^2 + z^2 = 3$$

is a sphere of radius $\sqrt{3}$ centered at the origin in $\mathbb{R}^3$. Level sets of the objective function

$$f(x, y, z) = x + 2y + 3z$$

are planes in R^3. The points a and b are the only two points on the sphere S at which one of these planes is tangent to the sphere. At the maximum point a, the gradient vectors ∇f and ∇g point in the same direction. At the minimum point b, ∇f and ∇g point in opposite directions.

Example 2.4. Maximize $x + 2y + 3z$ over the set $x^2 + y^2 + z^2 = 3$ and x =1.

The objective function is

$$f(x, y, z) = x + 2y + 3z,$$

so

$$\nabla f = (1,\, 2,\, 3).$$

The constraint functions are

$$\begin{aligned} g_1(x,\, y,\, z) &= x^2 + y^2 + z^2 \\ g_2(x,\, y,\, z) &= x. \end{aligned}$$

Compute

$$\begin{aligned} \nabla g_1 &= (2x,\, 2y,\, 2z) \\ \nabla g_2 &= (1,\, 0,\, 0). \end{aligned}$$

Then the Lagrange multiplier formula $\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2$ yields

$$\begin{aligned} 1 &= 2x\lambda_1 + \lambda_2 \\ 2 &= 2y\lambda_1 \\ 3 &= 2z\lambda_1. \end{aligned}$$

Solving for $x,\ y,$ and z in terms of λ_1 and λ_2 gives

$$\begin{aligned} x &= \frac{1-\lambda_2}{2\lambda_1} \\ y &= \frac{2}{2\lambda_1} \\ z &= \frac{3}{2\lambda_1}. \end{aligned}$$

Substituting into the constraint equation $x = 1$ gives $\lambda_2 = 1 - 2\lambda_1$. Substituting all of this into the remaining equation

$$x^2 + y^2 + z^2 = 3$$

yields a quadratic equation in λ_1, which gives $\lambda_1 = \pm\sqrt{26}/4$. Substituting back into the equations for x, y, and z yields the two following solutions:

$$\begin{aligned} c &= \left(1,\, \frac{2\sqrt{26}}{13},\, \frac{3\sqrt{26}}{13}\right) \\ d &= \left(1,\, \frac{-2\sqrt{26}}{13},\, \frac{-3\sqrt{26}}{13}\right). \end{aligned}$$

Since the two gradient vectors ∇g_1 and ∇g_2 are linearly independent everywhere on the constraint set, the points c and d are the only candidates for a maximum. Since f must attain its maximum on this closed and bounded set, we need only evaluate f at each candidate point to find the maximum. The maximum is

$$f(c) = 1 + \sqrt{26},$$

and the point d is the location of the minimum. The constraint set S in this example is a circle in $\mathbb{R}^3$ formed by the intersection of the sphere

$$x^2 + y^2 + z^2 = 3$$

and the plane $x = 1$. As before, the level sets of the function f are planes in $\mathbb{R}^3$. At the points c and d these planes are tangent to the circle S.

Inequality constraints can be handled by a combination of the Lagrange multiplier technique and the techniques for unconstrained problems. Suppose that the problem in Example 2.4 is altered by replacing the $x = 1$ constraint with the inequality constraint $x \geq 1$. We can consider the set

$$S = \{(x,\, y,\, z) : x^2 + y^2 + z^2 = 3,\ x \geq 1\}$$

as the union of two components. The maximum over the first component

$$S_1 = \{(x,\, y,\, z) : x^2 + y^2 + z^2 = 3,\ x = 1\}$$

was found to occur at the point

$$c = \left(1,\, \sqrt{\frac{8}{13}},\, 1.5\sqrt{\frac{8}{13}}\right)$$

in our previous analysis, and we can calculate that

$$f(x,\, y,\, z) = 1 + 6.5\sqrt{\frac{8}{13}} = 6.01$$

at this point. To consider the remaining part

$$S_2 = \{(x,\, y,\, z) : x^2 + y^2 + z^2 = 3,\ x > 1\},$$

we return to our analysis from Example 2.3, noting that there is no local maximum of f anywhere on this set. Therefore, the maximum of f on S_1 must be the maximum of the function f on the set S. If we had considered the maximum of f over the set

$$S = \{(x, y, z) : x^2 + y^2 + z^2 = 3,\ x \leq 1\},$$

then the maximum would be at the point

$$a = \left(\frac{1}{2},\, \frac{2}{2},\, \frac{3}{2}\right) \cdot \sqrt{\frac{6}{7}}$$

found in our analysis of Example 2.3.

Returning now to the problem introduced at the beginning of this section, we are ready to continue the modeling process with step 3. We will formulate the revised color TV problem as a constrained multivariable optimization problem. We wish to maximize $y = P$ (profit) as a function of our two decision variables, $x_1 = s$ and $x_2 = t$. We have the same objective function

$$\begin{aligned} y &= f(x_1, x_2) \\ &= (339 - 0.01x_1 - 0.003x_2)x_1 + (399 - 0.004x_1 - 0.01x_2)x_2 \\ &\quad - (400,000 + 195x_1 + 225x_2). \end{aligned}$$

We wish to maximize f over the set S consisting of all x_1 and x_2 satisfying the constraints

$$\begin{aligned} x_1 &\le 5,000 \\ x_2 &\le 8,000 \\ x_1 + x_2 &\le 10,000 \\ x_1 &\ge 0 \\ x_2 &\ge 0. \end{aligned}$$

The set S is called the *feasible region* because it represents the set of all feasible production levels. Figure 2.10 shows a graph of the feasible region for this problem.

We will apply Lagrange multiplier methods to find the maximum of $y = f(x_1, x_2)$ over the set S. Compute

$$\nabla f = (144 - 0.02x_1 - 0.007x_2,\ 174 - 0.007x_1 - 0.02x_2).$$

Since $\nabla f \neq 0$ in the interior of S, the maximum must occur on the boundary. Consider first the segment of the boundary on the constraint line

$$g(x_1, x_2) = x_1 + x_2 = 10,000.$$

Here $\nabla g = (1, 1)$, so the Lagrange multiplier equations are

$$\begin{aligned} 144 - 0.02x_1 - 0.007x_2 &= \lambda \\ 174 - 0.007x_1 - 0.02x_2 &= \lambda. \end{aligned} \tag{2.12}$$

Solving these two equations together with the constraint equation

$$x_1 + x_2 = 10,000$$

yields

$$\begin{aligned} x_1 &= \frac{50,000}{13} \approx 3,846 \\ x_2 &= \frac{80,000}{13} \approx 6,154 \\ \lambda &= 24. \end{aligned}$$

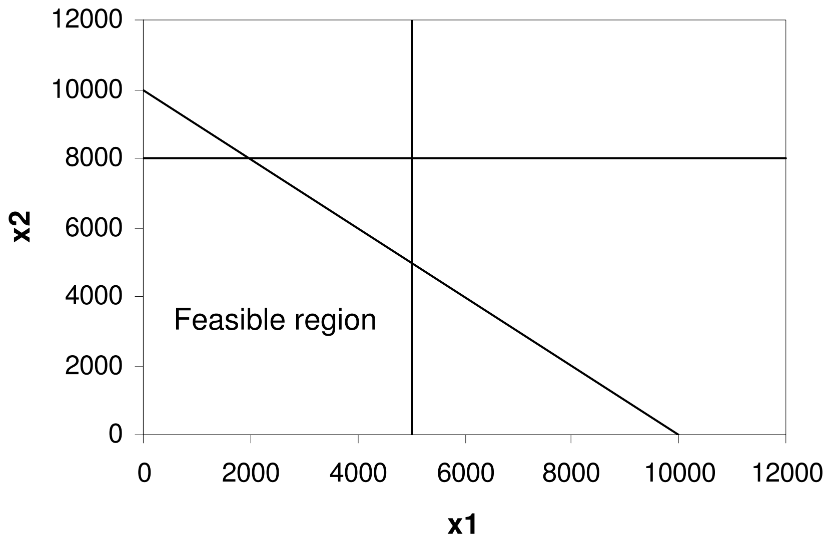

Figure 2.10: Graph showing the set of all feasible production levels x_1 of 19–inch sets and x_2 of 21–inch sets for the color TV problem with constraints.

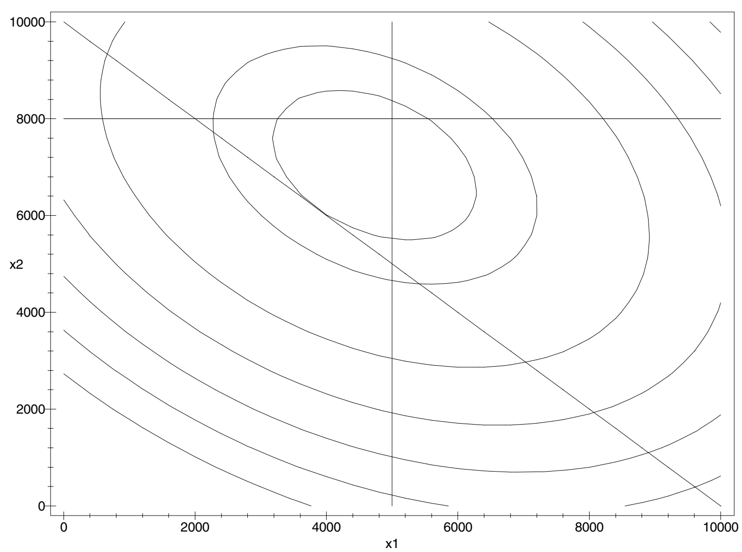

Figure 2.11: Graph showing level sets of profit $y = f(x_1, x_2)$ versus production levels x_1 of 19–inch sets and x_2 of 21–inch sets together with the set of all feasible production levels for the color TV problem with constraints.

Substituting back into Eq. (2.2), we obtain $y = 532,308$ at the maximum.

Figure 2.11 shows a Maple graph of the level sets of f together with the feasible region.

The level sets $f = C$ for $C = 0, 100,000, \ldots, 500,000$ form smaller and smaller concentric rings, all of which intersect the feasible region. The level set $f = 532,308$ forms the smallest ring. This set barely touches the feasible region S, and is tangent to the line $x_1 + x_2 = 10,000$ at the optimum point. This graphical evidence indicates that the critical point found by using Lagrange multipliers along the constraint line $x_1 + x_2 = 10,000$ is actually the maximum of the function f over the feasible region S.

```
In[1]:= y = (339 - x1/100 - 3 x2/1000) x1 +
          (399 - 4 x1/1000 - x2/100) x2 -
          (400000 + 195 x1 + 225 x2)
```

Out[1]= $\left(-\frac{x1}{100} - \frac{3\,x2}{1000} + 339\right)x1 - 195\,x1 + \left(-\frac{x1}{250} - \frac{x2}{100} + 399\right)x2 - 225\,x2 - 400000$

```
In[2]:= dydx1 = D[y, x1]
```

Out[2]= $-\frac{x1}{50} - \frac{7\,x2}{1000} + 144$

```
In[3]:= dydx2 = D[y, x2]
```

Out[3]= $-\frac{7\,x1}{1000} - \frac{x2}{50} + 174$

```
In[4]:= s = Solve[{dydx1 == lambda, dydx2 == lambda, x1 + x2 == 10000}, {x1, x2, lambda}]
```

Out[4]= $\left\{\left\{x1 \to \frac{50000}{13}, x2 \to \frac{80000}{13}, \text{lambda} \to 24\right\}\right\}$

```
In[5]:= N[%]

Out[5]= {{x1 → 3846.15, x2 → 6153.85, lambda → 24.}}

In[6]:= y /. %

Out[6]= {532308.}
```

Figure 2.12: Optimal solution to the color TV problem with constraints using the computer algebra system Mathematica.

An algebraic proof that this point is actually the maximum is a bit more complicated. By comparing values of f at this critical point with values at the endpoints $(5,000, 5,000)$ and $(2,000, 8,000)$, we can show that this critical point is the maximum over this line segment. Then we can optimize f over the remaining line segments and compare results. For example, the maximum of f over the line segment along the x_1 axis occurs at $x_1 = 5,000$. To see this, apply Lagrange multipliers with $g(x_1, x_2) = x_2 = 0$. Here $\nabla g = (0, 1)$, so the Lagrange multiplier equations are

$$\begin{aligned} 144 - 0.02x_1 - 0.007x_2 &= 0 \\ 174 - 0.007x_1 - 0.02x_2 &= \lambda. \end{aligned}$$

Solving these two equations together with the constraint equation $x_2 = 0$ yields $x_1 = 7,200$, $x_2 = 0$, and $\lambda = 123.6$. This is outside the feasible region, so the max–min along this segment must occur at the endpoints $(0, 0)$ and $(5,000, 0)$. The first is the minimum and the second is the maximum, since the value of f at the second is greater. It is also possible to optimize along this line segment by substituting $x_2 = 0$ and using one variable methods. Since the largest value of f occurs on the slanted line segment, we have found the maximum over S. Some of the calculations in step 4 are rather involved. In such cases it is appropriate to use a computer algebra system to simplify the process of computing derivatives and solving equations. Figure 2.12 shows the results of using the computer algebra system Mathematica to perform the calculations of step 4 for the constraint line $x_1 + x_2 = 10,000$.

In plain English, the company can maximize profits by producing 3,846 of the 19–inch sets and 6,154 of the 21–inch sets for a total of 10,000 sets per year. This level of production uses all of the available excess production capacity. The resource constraints on the availability of TV circuit boards are not binding. This venture will produce an estimated profit of \$532,308 annually.

2.3 Sensitivity Analysis and Shadow Prices

In this section we discuss some of the specialized techniques for sensitivity analysis in Lagrange multiplier models. It turns out that the multipliers themselves have a real–world significance.

Before we report the results of our model analysis in Example 2.2, it is important to perform sensitivity analysis. At the end of Section 2.1 we investigated the sensitivity to price elasticity for a model without constraints. The procedure for our new model is not much different. We examine the sensitivity to a particular parameter value by generalizing the model slightly, replacing the assumed value with a variable. Suppose we want to look again at the price elasticity, a, for 19–inch sets. We rewrite the objective function as in Eq. (2.7) so that

$$\nabla f = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2} \right),$$

where $\partial f / \partial x_1$ and $\partial f / \partial x_2$ are given by Eq. (2.8). Now the Lagrange multiplier equations are

$$\begin{aligned} 144 - 2ax_1 - 0.007x_2 &= \lambda \\ 174 - 0.007x_1 - 0.02x_2 &= \lambda. \end{aligned} \tag{2.13}$$