Root finding and solutions to non-linear equations (Taylor series for analytical functions, bisection, fixed point iteration, Newton's method and roots of polynomials)

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Contenido

Introduction problems

Determine the diameter of the hook to hang a hammock.

$$n = \frac{S_y}{\sigma}$$

$$\left(\frac{F_x}{\frac{\pi d^2}{4}} + \frac{(F_y L) d/2}{\frac{\pi d^4}{64}} \right) - \frac{Sy}{n} = 0$$

The question here is: **What method has the best performance doing this job?**

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Mathematical background

Errors

- True errors

$$E_t = \text{true value - approximation}$$

- True relative errors

$$E_r = \frac{\text{true value - approximation}}{\text{true value}}$$

- True relative errors

$$E_r^2 = \frac{(\text{true value - approximation})^2}{\text{true value}^2}$$

- Aproximate relative errors

$$A_r = \frac{\text{present approximation - previous approximation}}{\text{present approximation}}$$

- **Stopping criterion**

When the prefered error has an acceptable value

Mathematical background

Errors

Imagine we are going to approximate π by chopping:

Pi approximation	E_t	E_r	E_r^2
3,000000000000000	$1{,}416 \times 10^{-1}$	$4{,}507 \times 10^{-2}$	$2{,}031 \times 10^{-3}$
3,100000000000000	$1{,}416 \times 10^{-1}$	$4{,}507 \times 10^{-2}$	$2{,}031 \times 10^{-3}$
3,140000000000000	$4{,}159 \times 10^{-2}$	$1{,}324 \times 10^{-2}$	$1{,}753 \times 10^{-4}$
3,142000000000000	$1{,}593 \times 10^{-3}$	$5{,}070 \times 10^{-4}$	$2{,}570 \times 10^{-7}$
3,141600000000000	$-4{,}073 \times 10^{-4}$	$-1{,}297 \times 10^{-4}$	$1{,}681 \times 10^{-8}$
3,141590000000000	$-7{,}346 \times 10^{-6}$	$-2{,}338 \times 10^{-6}$	$5{,}468 \times 10^{-12}$
3,141593000000000	$2{,}654 \times 10^{-6}$	$8{,}447 \times 10^{-7}$	$7{,}135 \times 10^{-13}$
3,141592700000000	$-3{,}464 \times 10^{-7}$	$-1{,}103 \times 10^{-7}$	$1{,}216 \times 10^{-14}$
3,141592650000000	$-4{,}641 \times 10^{-8}$	$-1{,}477 \times 10^{-8}$	$2{,}182 \times 10^{-16}$
3,141592654000000	$3{,}590 \times 10^{-9}$	$1{,}143 \times 10^{-9}$	$1{,}306 \times 10^{-18}$
3,141592653600000	$-4{,}102 \times 10^{-10}$	$-1{,}306 \times 10^{-10}$	$1{,}705 \times 10^{-20}$
3,141592653590000	$-1{,}021 \times 10^{-11}$	$-3{,}249 \times 10^{-12}$	$1{,}056 \times 10^{-23}$
3,141592653590000	$-2{,}069 \times 10^{-13}$	$-6{,}587 \times 10^{-14}$	$4{,}339 \times 10^{-27}$
3,141592653589800	$-2{,}069 \times 10^{-13}$	$-6{,}587 \times 10^{-14}$	$4{,}339 \times 10^{-27}$
3,141592653589790	$-7{,}105 \times 10^{-15}$	$-2{,}262 \times 10^{-15}$	$5{,}115 \times 10^{-30}$
3,141592653589793	$3{,}109 \times 10^{-15}$	$9{,}895 \times 10^{-16}$	$9{,}791 \times 10^{-31}$

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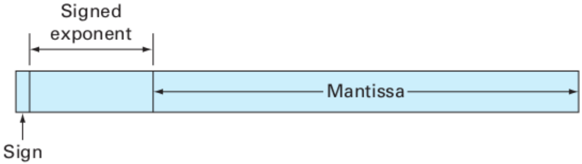
Mathematical background

Approximations and Round-Off Errors

Given the fact that on computers only a fixed number of significan digits can be assigned to represent a number, numbers with more significant digits of the max in the system should be round-off to the better approximation in the system.



This arrives with several problems:

- The division by a very small number or multiplication by a very large number
- Adding a large and a small number
- Subtractive Cancellation

We can avoid this problems

⋆ Reducing the operations number

⋆ Planning the operations order

⋆ Restating the problem

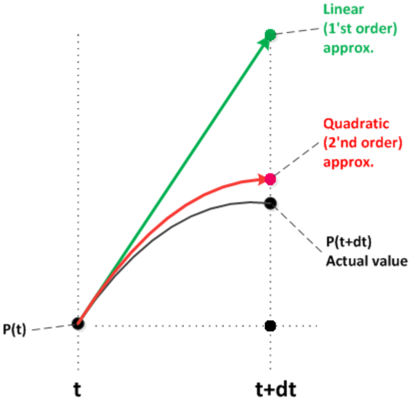
Mathematical background

Truncation Errors

Taylor Series

$$f(x_{i+1}) = f(x_i) + f'(x_i) * h + \frac{f''(x_i) * h^2}{!2} + \cdots + \frac{f^{(n)}(x_i) * h^n}{!n} + O(h^{n+1})$$

Provides a means to predict a function value at one point in terms of the function value and its derivatives at another point. It is impossible, or practically impossible, the using the exact formulation because is a infinite series. To apply it we have to truncate the series and work with the resulting expression.



It is used to

- Estimate numerical errors in another numerical approximations
- To evaluate derivatives (Finite Diference Method)

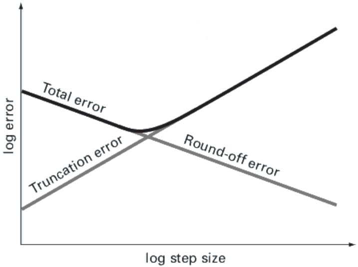
Mathematical background

Total numerical erros

Error propagation

$$\Delta F(\vec{x}) = \sum_{i=1}^{n} \left| \frac{\partial F}{\partial x_i} \right| \Delta x_i$$

Total error



The total numerical error is the summation of the round-off and truncation errors.

Root finding and solutions to non-linear equations

Bisection

Based on the **Mean and Intermedia value theorems**, this method take advantage of the change the sing of function at the root.

Step 1: Choose lower x_l and upper x_u guesses for the root such that the function changes sign over the interval. This can be checked by ensuring that $f(x_l)f(x_u) < 0$.

Step 2: An estimate of the root x_r is determined by

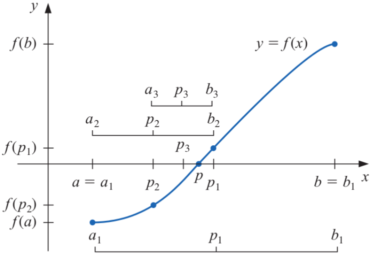
$$x_r = \frac{x_l + x_u}{2}$$

Step 3: Make the following evaluations to determine in which subinterval the root lies:

(a) If $f(x_l)f(x_r) < 0$, the root lies in the lower subinterval. Therefore, set $x_u = x_r$ and return to step 2.

(b) If $f(x_l)f(x_r) > 0$, the root lies in the upper subinterval. Therefore, set $x_l = x_r$ and return to step 2.

(c) If $f(x_l)f(x_r) = 0$, the root equals x_r; terminate the computation.



Stop criterion: When the values of a_n, b_n, a_n be near to 0 or very close together.

Iterations number: If we have a desired approximate error A_t^d, we can evaluate the iteration number as

$$n = \log_2\left(\frac{b_1 - a_1}{A_t^d}\right)$$

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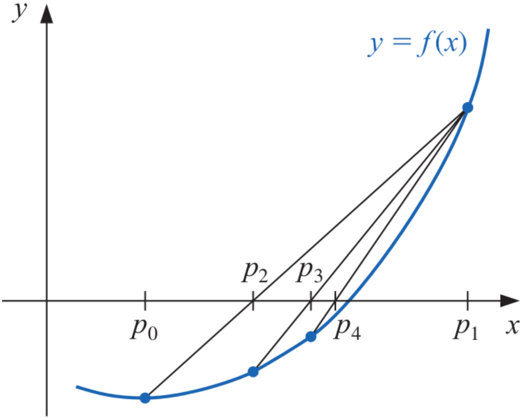
Root finding and solutions to non-linear equations

Regula Falsi

This method can be saw like a modification of last method. Before we use only the sign of the limits of interval but not their value. Regula Falsi method use this values to predict the point to generate the new interval.

The steps are the same of bisection method, but in the second one we need rewrite the expression to calculate the new point.

$$x_r = x_u - f(x_u)\frac{x_l - x_u}{f(x_l) - f(x_u)}$$



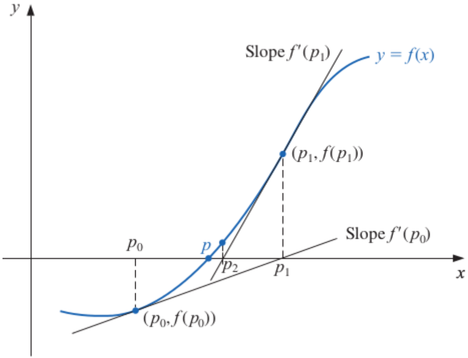
Stop criterion: When the values of a_n, b_n, a_n be near to 0 or very close together.

Root finding and solutions to non-linear equations

Newton Raphson

We suppose that f is a C^2 function on a given interval, then using Taylor's expansion near x we can formulate the next serie and if x_0 is enought close to a root, the serie converge to this root.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$



Root finding and solutions to non-linear equations

Newton Raphson caracteristics

- When does the method converges it doing quickly. Cuadratic convergence
- Presents troubles when $f'(x)$ is near to 0
- Is necessary know of $x_0, f(x_i), f'(x_i)$. Secant and Constant scope methods.

  Aproximate the derivatives using a secant line or assume a constant scope to make the next aproximations
- The convergence is slow in multiple roots. Modified Newton Raphson Method

  We have to make a auxiliar function $u(x) = f(x)/f'(x)$ and we apply the method to this function. This mean resolve

$$x_{i+1} = x_i - \frac{f(x_i)f'(x_i)}{[f'(x_i)]^2 - f(x_i)f'(x_i)}$$

Root finding and solutions to non-linear equations

Newton Raphson generalization

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$[x]_{n+1} = [x]_n - [J([x]_n)]^{-1} f([x]_n)$$

where:

$$J = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial x_1} & \cdots & \frac{\partial F_m}{\partial x_n} \end{bmatrix}$$

For practical pourporse is better rewrite that equation in the next form.

$$[h]_n = [J([x]_n)] \backslash f([x]_n)$$

$$[x]_{n+1} = [x]_n - [h]_n$$

Homework

Some considerations about of homework:

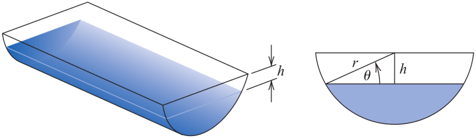
- The homework must be presented in a format used to make articles of IEEE. You can found in the course site or use the IEEETRAN class in latex
- It is necessary presented an *Introduction* to the problem to work and some *Mathematical background* **USED** in the developed of the work, then the *Developed of the work*, next a serie of *Conclusions* and finally *References*.

Homework

1. We consider the bisection method to find a root for $f(x)$. If $f(0) < 0$ and $f(1) > 0$, How many steps of the bisection method are needed to obtain an approximation to the root, if the absolute error is $\leq 10^{-6}$? What cosiderations you can say about the function $f(x)$ and the its relationship with the number of steps to obtain this error?
2. The function $f(x) = \frac{4x-7}{x-2}$ using the Newton method and its modifications plot convergence graph for the following initial conditions to approximate p. $x_0 = [1{,}625; 1{,}875; 1{,}5; 1{,}95]$
3. The function $f(x) = (x-1)^m$ using the Newton method and Modified Newton Raphson plot convergence graph and analyze the behavior when m change
4. Implement several numerical methods to obtain the result of $\sqrt[3]{30}$. Which is better?

Homework

5. A trough of length L has a cross section in the shape of a semicircle with radius r. Find a the relation between V and h and using this equation with $L = 10\text{ft}$, $r = 1\text{ft}$, and $V = 12{,}4\text{ft}^3$ find the value of h through numerical methods.



6. Try to speed up the bisection method spliting the domain in 3 interval, using the Regula Falsi expresion and select the smalest interval for the next iteration. Argue the behavior with the convergence graph.
7. Solve the next systems of nonlinear equations using Newton Raphson method.
   - 7.1 $x^2 + y^2 = 2$; $x + y = 2$
   - 7.2 $x^2 - xy + 3y^2 = 27$; $x - y = 2$
   - 7.3 $y = \sqrt(x)$; $(x + 2)^2 + y^2 = 1$