Fórmulas de INTEGRACIÓN CERRADA de NEWTON-COTES

$$m = 1 \longrightarrow \int_{x_0}^{x_1} f(x)dx = \frac{h}{2}(f_0 + f_1) - \frac{h^3}{12} f^{(2)}(\xi) \qquad \textbf{Regla del Trapecio}$$

$$m = 2 \longrightarrow \int_{x_0}^{x_2} f(x)dx = \frac{h}{3}((f_0 + f_2) + 4f_1) - \frac{h^5}{90} f^{(4)}(\xi) \qquad \textbf{Regla de Simpson}$$

$$m = 3 \longrightarrow \int_{x_0}^{x_3} f(x)dx = \frac{3h}{8}((f_0 + f_3) + 3(f_1 + f_2)) - \frac{3}{80}\, h^5 f^{(4)}(\xi) \qquad \textbf{Segunda Regla de Simpson}$$

$$m = 4 \longrightarrow \int_{x_0}^{x_4} f(x)dx = \frac{2h}{45}(7(f_0 + f_4) + 32(f_1 + f_3) + 12f_2) - \frac{8}{945}\, h^7 f^{(6)}(\xi) \qquad \textbf{Regla de Bode}$$

$$m = 5 \longrightarrow \int_{x_0}^{x_5} f(x)dx = \frac{5h}{288}(19(f_0 + f_5) + 75(f_1 + f_4) + 50(f_2 + f_3)) - \frac{275}{12096}\, h^7 f^{(6)}(\xi)$$

$$m = 6 \longrightarrow \int_{x_0}^{x_6} f(x)dx = \frac{h}{140}(41(f_0 + f_6) + 216(f_1 + f_5) + 27(f_2 + f_4) + 272f_3) - \frac{9}{1400}\, h^9 f^{(8)}(\xi)$$

$$m = 7 \longrightarrow \int_{x_0}^{x_7} f(x)dx = \frac{7h}{17280}(751(f_0 + f_7) + 3577(f_1 + f_6) + 1323(f_2 + f_5) + 2989(f_3 + f_4)) - \frac{8183}{518400}\, h^9 f^{(8)}(\xi)$$

$$m = 8 \longrightarrow \int_{x_0}^{x_8} f(x)dx = \frac{4h}{14175}(989(f_0 + f_8) + 5888(f_1 + f_7) - 928(f_2 + f_6) + 10496(f_3 + f_5) - 4540f_4) - \\ - \frac{2368}{467775}\, h^{11} f^{(10)}(\xi)$$

$$m = 9 \longrightarrow \int_{x_0}^{x_9} f(x)dx = \frac{9h}{89600}(2857(f_0 + f_9) + 15741(f_1 + f_8) + 1080(f_2 + f_7) + 19344(f_3 + f_6) + \\ + 5778(f_4 + f_5)) - \frac{173}{14620}\, h^{11} f^{(10)}(\xi)$$

$$m = 10 \longrightarrow \int_{x_0}^{x_{10}} f(x)dx = \frac{5h}{299376}(16067(f_0 + f_{10}) + 106300(f_1 + f_9) - 48525(f_2 + f_8) + 272400(f_3 + f_7) + \\ - 260550(f_4 + f_6) + 427368f_5) - \frac{1346350}{326918592}\, h^{13} f^{(12)}(\xi)$$

Fórmulas de INTEGRACIÓN ABIERTA de NEWTON-COTES

$$m = 0 \longrightarrow \int_a^b f(x)dx = 2h\, f_0 + \frac{h^3}{3} f^{(2)}(\xi)$$

$$m = 1 \longrightarrow \int_a^b f(x)dx = \frac{3h}{2}(f_0 + f_1) + \frac{h^3}{4} f^{(2)}(\xi)$$

$$m = 2 \longrightarrow \int_a^b f(x)dx = \frac{4h}{3}(2(f_0 + f_2) - f_1) + \frac{28}{90}\, h^5 f^{(4)}(\xi)$$

$$m = 3 \longrightarrow \int_a^b f(x)dx = \frac{5h}{24}(11(f_0 + f_3) + (f_1 + f_2)) + \frac{95}{144}\, h^5 f^{(4)}(\xi)$$

$$m = 4 \longrightarrow \int_a^b f(x)dx = \frac{6h}{20}(11(f_0 + f_4) - 14(f_1 + f_3) + 26f_2) + \frac{41}{140}\, h^7 f^{(6)}(\xi)$$

$$m = 5 \longrightarrow \int_a^b f(x)dx = \frac{7h}{1440}(611(f_0 + f_5) - 453(f_1 + f_4) + 562(f_2 + f_3)) + \frac{5257}{8640}\, h^7 f^{(6)}(\xi)$$

$$m = 6 \longrightarrow \int_a^b f(x)dx = \frac{8h}{945}(460(f_0 + f_6) - 954(f_1 + f_5) + 2196(f_2 + f_4) - 2459f_3) + \frac{3956}{14175}\, h^9 f^{(8)}(\xi)$$

Fórmulas de INTEGRACIÓN COMPUESTAS

Fórmula Compuesta del Trapecio

$$\int_a^b f(x)dx = \frac{h}{2}\Big(f_0 + 2f_1 + 2f_2 + \ldots + 2f_{n-1} + f_n\Big) - \frac{(b-a)^3}{12n^2} f^{(2)}(\xi); \qquad h = \frac{b-a}{n}$$

Fórmula Compuesta del Simpson

$$\int_a^b f(x)dx = \frac{h}{3}\Big(f_0 + 4f_1 + 2f_2 + 4f_3 + \ldots + 2f_{2n-2} + 4f_{2n-1} + f_{2n}\Big) - \frac{(b-a)^5}{2880n^4} f^{(4)}(\xi); \qquad h = \frac{b-a}{2n}$$

Extrapolación de RICHARDSON

Sea I_1 el valor de la integral calculada con n_1 puntos y sea I_2 el valor de la integral calculada con n_2 puntos. La combinación de fórmulas de integración compuestas con la extrapolación de Richardson conduce a una nueva aproximación a la integral I_R:

Fórmula Compuesta del Trapecio: $$I_R = \frac{I_2(n_2/n_1)^2 - I_1}{(n_2/n_1)^2 - 1}. \quad \text{Si } n_2 = 2n_1, \text{ entonces } I_R = \frac{4I_2 - I_1}{3}$$

Fórmula Compuesta de Simpson: $$I_R = \frac{I_2(n_2/n_1)^4 - I_1}{(n_2/n_1)^4 - 1}. \quad \text{Si } n_2 = 2n_1, \text{ entonces } I_R = \frac{16I_2 - I_1}{15}$$

Método de Integración de ROMBERG

El algoritmo del método de integración de Romberg combina la fórmula de integración compuesta del trapecio con la extrapolación de Richardson.

Fase 1)

$$T_{0,1} = \frac{b-a}{2}\Big(f(a) + f(b)\Big)$$

$$T_{i,1} = \frac{1}{2}\left(T_{i-1,1} + h\sum_{p=0}^{P-1} f(x_p)\right) \quad \text{siendo } x_p = a + \left(\frac{2p+1}{2}\right)h; \quad h = \frac{b-a}{P}; \quad P = 2^{i-1}$$

Fase 2)

$$T_{i,j} = \frac{4^{j-1}T_{i+1,j-1} - T_{i,j-1}}{4^{j-1} - 1}; \quad i = 0, N-j+1; \; j = 0, N+1.$$

Cada fila y columna de la matriz que se puede construir con los coeficientes $T_{i,j}$ converge al valor de la integral

$$I = \int_a^b f(x)dx.$$

El error de integración es de la forma general

$$\varepsilon_{i,j} = \frac{constante(a,b,j)}{2^{2ji}} f^{(2j)}(\xi), \qquad \xi \in [a,b]$$

COMPARACIÓN DE CUADRATURAS DE NEWTON-COTES Y FÓRMULAS COMPUESTAS

El Ejemplo consiste en evaluar numéricamente mediante Cuadraturas de Newton-Cotes de distinto número de puntos y con las fórmulas compuestas del Trapecio y de Simpson el valor de la integral:

$$\int_{-4}^{+4} \frac{1}{1+x^2} dx$$

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***************************
Cuadraturas de NEWTON-COTES
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Puntos	Val. Exac.	Val. Aprox	Err. Rel.(%)
2	2.6516353273	0.4705882353	82.252905200
3	2.6516353273	5.4901960784	-107.049439334
4	2.6516353273	2.2776470588	14.104061168
5	2.6516353273	2.2776470588	14.104061168
6	2.6516353273	2.3722292496	10.537123067
7	2.6516353273	3.3287981275	-25.537553869
8	2.6516353273	2.7997007825	-5.583929797
9	2.6516353273	1.9410943044	26.796332649
10	2.6516353273	2.4308411566	8.326717042

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*********************************************
Formulas Compuestas del Trapecio y Simpson 1/3
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		Fórmula Compuesta del Trapecio			**Fórmula Compuesta Simpson 1/3**		
Intervalos	Val. Exac.	Puntos	Val. Aprox	Err. Rel.(%)	Puntos	Val. Aprox	Err. Rel.(%)
1	2.6516353273	2	0.4705882353	82.252905200	3	5.4901960784	-107.049439334
2	2.6516353273	3	4.2352941176	-59.723853201	5	2.4784313725	6.531967386
3	2.6516353273	4	2.0768627451	21.676154949	7	2.9084215239	-9.684069068
4	2.6516353273	5	2.9176470588	-10.031987760	9	2.5725490196	2.982548426
5	2.6516353273	6	2.5187099403	5.012958820	11	2.6952859224	-1.646176402
6	2.6516353273	7	2.7005318292	-1.844013064	13	2.6332910535	0.691809828
7	2.6516353273	8	2.6200579018	1.190866075	15	2.6602997673	-0.326758355
8	2.6516353273	9	2.6588235294	-0.271085620	17	2.6477345635	0.147107854
9	2.6516353273	10	2.6426739520	0.337956553	19	2.6534158938	-0.067149749
10	2.6516353273	11	2.6511419269	0.018607403	21	2.6508184459	0.030806705
11	2.6516353273	12	2.6480963324	0.133464618	23	2.6520029475	-0.013863904
12	2.6516353273	13	2.6501012474	0.057854105	25	2.6514640295	0.006460083
13	2.6516353273	14	2.6496637693	0.074352533	27	2.6517107066	-0.002842745
14	2.6516353273	15	2.6502393009	0.052647752	29	2.6515989345	0.001372466
15	2.6516353273	16	2.6502787921	0.051158439	31	2.6516503898	-0.000568044
16	2.6516353273	17	2.6505068050	0.042559485	33	2.6516272830	0.000303374
17	2.6516353273	18	2.6506059693	0.038819742	35	2.6516380757	-0.000103647
18	2.6516353273	19	2.6507304083	0.034126827	37	2.6516333457	0.000074734
19	2.6516353273	20	2.6508168216	0.030867960	39	2.6516356461	-0.000012020
20	2.6516353273	21	2.6508993161	0.027756879	41	2.6516347072	0.000023386
25	2.6516353273	26	2.6511633755	0.017798521	51	2.6516352085	0.000004480
30	2.6516353273	31	2.6513074904	0.012363577	61	2.6516352668	0.000002281
40	2.6516353273	41	2.6514508595	0.006956759	81	2.6516353082	0.000000721
50	2.6516353273	51	2.6515172503	0.004452990	101	2.6516353195	0.000000296
100	2.6516353273	101	2.6516058022	0.001113469	201	2.6516353268	0.000000018
200	2.6516353273	201	2.6516279457	0.000278381	401	2.6516353273	0.000000001
400	2.6516353273	401	2.6516334819	0.000069596	801	2.6516353273	0.000000000
800	2.6516353273	801	2.6516348660	0.000017399	1601	2.6516353273	0.000000000

Cuadratura de GAUSS-LEGENDRE

$$\int_{-1}^{+1} f(z)dz \approx \sum_{i=0}^{i=n} \omega_i\, f(z_i) \qquad \int_a^b F(x)dx \approx \frac{b-a}{2} \sum_{i=0}^{i=n} \omega_i\, F\left(\frac{(b-a)z_i+(b+a)}{2}\right)$$

n	i	z_i	ω_i
0	0	0.00000000000000E+00	0.20000000000000E+01
1	0	-0.57735026918963E+00	0.10000000000000E+01
	1	0.57735026918963E+00	0.10000000000000E+01
2	0	-0.77459666924148E+00	0.55555555555556E+00
	1	0.00000000000000E+00	0.88888888888889E+00
	2	0.77459666924148E+00	0.55555555555556E+00
3	0	-0.86113631159405E+00	0.34785484513745E+00
	1	-0.33998104358486E+00	0.65214515486255E+00
	2	0.33998104358486E+00	0.65214515486255E+00
	3	0.86113631159405E+00	0.34785484513745E+00
4	0	-0.90617984593866E+00	0.23692688505619E+00
	1	-0.53846931010568E+00	0.47862867049937E+00
	2	0.00000000000000E+00	0.56888888888889E+00
	3	0.53846931010568E+00	0.47862867049937E+00
	4	0.90617984593866E+00	0.23692688505619E+00
5	0	-0.93246951420315E+00	0.17132449237917E+00
	1	-0.66120938646626E+00	0.36076157304814E+00
	2	-0.23861918608320E+00	0.46791393457269E+00
	3	0.23861918608320E+00	0.46791393457269E+00
	4	0.66120938646626E+00	0.36076157304814E+00
	5	0.93246951420315E+00	0.17132449237917E+00
6	0	-0.94910791234276E+00	0.12948496616887E+00
	1	-0.74153118559939E+00	0.27970539148928E+00
	2	-0.40584515137740E+00	0.38183005050512E+00
	3	0.00000000000000E+00	0.41795918367347E+00
	4	0.40584515137740E+00	0.38183005050512E+00
	5	0.74153118559939E+00	0.27970539148928E+00
	6	0.94910791234276E+00	0.12948496616887E+00
7	0	-0.96028985649754E+00	0.10122853629038E+00
	1	-0.79666647741363E+00	0.22238103445337E+00
	2	-0.52553240991633E+00	0.31370664587789E+00
	3	-0.18343464249565E+00	0.36268378337836E+00
	4	0.18343464249565E+00	0.36268378337836E+00
	5	0.52553240991633E+00	0.31370664587789E+00
	6	0.79666647741363E+00	0.22238103445337E+00
	7	0.96028985649754E+00	0.10122853629038E+00
9	0	-0.97390652851717E+00	0.66671344308688E-01
	1	-0.86506336668898E+00	0.14945134915058E+00
	2	-0.67940956829902E+00	0.21908636251598E+00
	3	-0.43339539412925E+00	0.26926671931000E+00
	4	-0.14887433898163E+00	0.29552422471475E+00
	5	0.14887433898163E+00	0.29552422471475E+00
	6	0.43339539412925E+00	0.26926671931000E+00
	7	0.67940956829902E+00	0.21908636251598E+00
	8	0.86506336668898E+00	0.14945134915058E+00
	9	0.97390652851717E+00	0.66671344308688E-01
11	0	-0.98156063424672E+00	0.47175336386511E-01
	1	-0.90411725637047E+00	0.10693932599532E+00
	2	-0.76990267419431E+00	0.16007832854335E+00
	3	-0.58731795428662E+00	0.20316742672307E+00
	4	-0.36783149899818E+00	0.23349253653835E+00
	5	-0.12523340851147E+00	0.24914704581340E+00
	6	0.12523340851147E+00	0.24914704581340E+00
	7	0.36783149899818E+00	0.23349253653835E+00
	8	0.58731795428662E+00	0.20316742672307E+00
	9	0.76990267419431E+00	0.16007832854335E+00
	10	0.90411725637047E+00	0.10693932599532E+00
	11	0.98156063424672E+00	0.47175336386511E-01

$$R_n = \left(\frac{b-a}{2}\right)^{2n+3} \frac{2^{2n+3}((n+1)!)^4}{(2n+3)((2n+2)!)^3}\, F^{(2n+2)}(\xi), \quad \xi \in [a,b]$$

Cuadratura de GAUSS-LAGUERRE

$$\int_0^\infty e^{-z} f(z)dz \approx \sum_{i=0}^{i=n} \omega_i f(z_i) \qquad \int_a^\infty F(x)dx \approx \sum_{i=0}^{i=n} \omega_i e^{z_i} F(a+z_i)$$

n	i	z_i	ω_i	$\omega_i e^{z_i}$
0	0	0.10000000000000E+01	0.10000000000000E+01	0.27182818284590E+01
1	0	0.58578643762690E+00	0.85355339059327E+00	0.15333260331194E+01
	1	0.34142135623731E+01	0.14644660940673E+00	0.44509573350546E+01
2	0	0.41577455678348E+00	0.71109300992917E+00	0.10776928592709E+01
	1	0.22942803602790E+01	0.27851773356924E+00	0.27621429619016E+01
	2	0.62899450829375E+01	0.10389256501586E-01	0.56010946254344E+01
3	0	0.32254768961939E+00	0.60315410434163E+00	0.83273912383789E+00
	1	0.17457611011583E+01	0.35741869243780E+00	0.20481024384543E+01
	2	0.45366202969211E+01	0.38887908515005E-01	0.36311463058215E+01
	3	0.93950709123011E+01	0.53929470556133E-03	0.64871450844077E+01
4	0	0.26356031971814E+00	0.52175561058281E+00	0.67909404220775E+00
	1	0.14134030591065E+01	0.39866681108318E+00	0.16384878736027E+01
	2	0.35964257710407E+01	0.75942449681708E-01	0.27694432423708E+01
	3	0.70858100058588E+01	0.36117586799221E-02	0.43156569009209E+01
	4	0.12640800844276E+02	0.23369972385776E-04	0.72191863543545E+01
5	0	0.22284660417926E+00	0.45896467394996E+00	0.57353550742274E+00
	1	0.11889321016726E+01	0.41700083077212E+00	0.13692525907123E+01
	2	0.29927363260593E+01	0.11337338207405E+00	0.22606845933827E+01
	3	0.57751435691045E+01	0.10399197453149E-01	0.33505245823555E+01
	4	0.98374674183826E+01	0.26101720281493E-03	0.48868268002108E+01
	5	0.15982873980602E+02	0.89854790642962E-06	0.78490159455958E+01
6	0	0.19304367656036E+00	0.40931895170127E+00	0.49647759753997E+00
	1	0.10266648953392E+01	0.42183127786172E+00	0.11776430608612E+01
	2	0.25678767449507E+01	0.14712634865751E+00	0.19182497816598E+01
	3	0.49003530845265E+01	0.20633514468717E-01	0.27718486362321E+01
	4	0.81821534445629E+01	0.10740101432807E-02	0.38412491224885E+01
	5	0.12734180291798E+02	0.15865464348564E-04	0.53806782079215E+01
	6	0.19395727862263E+02	0.31703154789956E-07	0.84054324868284E+01
7	0	0.17027963230510E+00	0.36918858934164E+00	0.43772341049291E+00
	1	0.90370177679938E+00	0.41878678081434E+00	0.10338693476656E+01
	2	0.22510866298661E+01	0.17579498663717E+00	0.16697097656588E+01
	3	0.42667001702877E+01	0.33343492261216E-01	0.23769247017586E+01
	4	0.70459054023935E+01	0.27945362352257E-02	0.32085409133479E+01
	5	0.10758516010181E+02	0.90765087733582E-04	0.42685755108251E+01
	6	0.15740678641278E+02	0.84857467162725E-06	0.58180833686719E+01
	7	0.22863131736889E+02	0.10480011748715E-08	0.89062262152922E+01
8	0	0.15232222773181E+00	0.33612642179796E+00	0.39143112431564E+00
	1	0.80722002274225E+00	0.41121398042399E+00	0.92180502852896E+00
	2	0.20051351556193E+01	0.19928752537089E+00	0.14801279099429E+01
	3	0.37834739733312E+01	0.47460562765652E-01	0.20867708075493E+01
	4	0.62049567778766E+01	0.55996266107946E-02	0.27729213897120E+01
	5	0.93729852516876E+01	0.30524976709321E-03	0.35916260680923E+01
	6	0.13466236911092E+02	0.65921230260753E-05	0.46487660021402E+01
	7	0.18833597788992E+02	0.41107693303496E-07	0.62122754197471E+01
	8	0.26374071890927E+02	0.32908740303507E-10	0.93632182377058E+01

$$R_n = \frac{((n+1)!)^2}{(2n+2)!}\, f^{(2n+2)}(\xi), \quad \xi \in [0,+\infty), \quad f(z) = e^{z} F(a+z)$$

Cuadratura de GAUSS-HERMITE

$$\int_{-\infty}^{\infty} e^{-z^2} f(z)dz \approx \sum_{i=0}^{i=n} \omega_i f(z_i) \qquad \int_{-\infty}^{\infty} F(x)dx \approx \sum_{i=0}^{i=n} \omega_i e^{z_i^2} F(z_i)$$

n	i	z_i	ω_i	$\omega_i e^{z_i^2}$
0	0	0.00000000000000E+00	0.17724538509055E+01	0.17724538509055E+01
1	0	-0.70710678118655E+00	0.88622692545276E+00	0.14611411826611E+01
	1	0.70710678118655E+00	0.88622692545276E+00	0.14611411826611E+01
2	0	-0.12247448713916E+01	0.29540897515092E+00	0.13239311752136E+01
	1	0.00000000000000E+00	0.11816359006037E+01	0.11816359006037E+01
	2	0.12247448713916E+01	0.29540897515092E+00	0.13239311752136E+01
3	0	-0.16506801238858E+01	0.81312835447245E-01	0.12402258176958E+01
	1	-0.52464762327529E+00	0.80491409000551E+00	0.10599644828950E+01
	2	0.52464762327529E+00	0.80491409000551E+00	0.10599644828950E+01
	3	0.16506801238858E+01	0.81312835447245E-01	0.12402258176958E+01
4	0	-0.20201828704561E+01	0.19953242059046E-01	0.11814886255360E+01
	1	-0.95857246461382E+00	0.39361932315224E+00	0.98658099675143E+00
	2	0.00000000000000E+00	0.94530872048294E+00	0.94530872048294E+00
	3	0.95857246461382E+00	0.39361932315224E+00	0.98658099675143E+00
	4	0.20201828704561E+01	0.19953242059046E-01	0.11814886255360E+01
5	0	-0.23506049736745E+01	0.45300099055088E-02	0.11369083326745E+01
	1	-0.13358490740137E+01	0.15706732032286E+00	0.93558055763118E+00
	2	-0.43607741192762E+00	0.72462959522439E+00	0.87640133443623E+00
	3	0.43607741192762E+00	0.72462959522439E+00	0.87640133443623E+00
	4	0.13358490740137E+01	0.15706732032286E+00	0.93558055763118E+00
	5	0.23506049736745E+01	0.45300099055088E-02	0.11369083326745E+01
6	0	-0.26519613568352E+01	0.97178124509952E-03	0.11013307296103E+01
	1	-0.16735516287675E+01	0.54515582819127E-01	0.89718460022519E+00
	2	-0.81628788285896E+00	0.42560725261013E+00	0.82868730328364E+00
	3	0.00000000000000E+00	0.81026461755681E+00	0.81026461755681E+00
	4	0.81628788285896E+00	0.42560725261013E+00	0.82868730328364E+00
	5	0.16735516287675E+01	0.54515582819127E-01	0.89718460022519E+00
	6	0.26519613568352E+01	0.97178124509952E-03	0.11013307296103E+01
7	0	-0.29306374202572E+01	0.19960407221137E-03	0.10719301442480E+01
	1	-0.19816567566958E+01	0.17077983007413E-01	0.86675260656338E+00
	2	-0.11571937124468E+01	0.20780232581489E+00	0.79289004838640E+00
	3	-0.38118699020732E+00	0.66114701255824E+00	0.76454412865173E+00
	4	0.38118699020732E+00	0.66114701255824E+00	0.76454412865173E+00
	5	0.11571937124468E+01	0.20780232581489E+00	0.79289004838640E+00
	6	0.19816567566958E+01	0.17077983007413E-01	0.86675260656338E+00
	7	0.29306374202572E+01	0.19960407221137E-03	0.10719301442480E+01
9	0	-0.34361591188377E+01	0.76404328552326E-05	0.10254516913657E+01
	1	-0.25327316742328E+01	0.13436457467812E-02	0.82066612640481E+00
	2	-0.17566836492999E+01	0.33874394455481E-01	0.74144193194356E+00
	3	-0.10366108297895E+01	0.24013861108231E+00	0.70329632310491E+00
	4	-0.34290132722370E+00	0.61086263373533E+00	0.68708185395127E+00
	5	0.34290132722370E+00	0.61086263373533E+00	0.68708185395127E+00
	6	0.10366108297895E+01	0.24013861108231E+00	0.70329632310491E+00
	7	0.17566836492999E+01	0.33874394455481E-01	0.74144193194356E+00
	8	0.25327316742328E+01	0.13436457467812E-02	0.82066612640481E+00
	9	0.34361591188377E+01	0.76404328552326E-05	0.10254516913657E+01
11	0	-0.38897248978698E+01	0.26585516843563E-06	0.98969904709229E+00
	1	-0.30206370251209E+01	0.85736870435878E-04	0.78664393946332E+00
	2	-0.22795070805011E+01	0.39053905846291E-02	0.70522036611222E+00
	3	-0.15976826351526E+01	0.51607985615884E-01	0.66266277326687E+00
	4	-0.94778839124016E+00	0.26049231026416E+00	0.63962123202026E+00
	5	-0.31424037625436E+00	0.57013523626248E+00	0.62930787436949E+00
	6	0.31424037625436E+00	0.57013523626248E+00	0.62930787436949E+00
	7	0.94778839124016E+00	0.26049231026416E+00	0.63962123202026E+00
	8	0.15976826351526E+01	0.51607985615884E-01	0.66266277326687E+00
	9	0.22795070805011E+01	0.39053905846291E-02	0.70522036611222E+00
	10	0.30206370251209E+01	0.85736870435878E-04	0.78664393946332E+00
	11	0.38897248978698E+01	0.26585516843563E-06	0.98969904709229E+00

$$R_n = \frac{\sqrt{\pi}\,(n+1)!}{2^{n+1}(2n+2)!}\, f^{(2n+2)}(\xi), \quad \xi \in (-\infty, +\infty), \quad f(z) = e^{z^2} F(z)$$

Cuadratura de GAUSS-TCHEBYSHEV

$$\int_{-1}^{+1} \frac{1}{\sqrt{1-z^2}} f(z)dz \approx \sum_{i=0}^{i=n} \omega_i\, f(z_i) \qquad \int_a^b F(x)dx \approx \frac{b-a}{2} \sum_{i=0}^{i=n} \omega_i \sqrt{1-z_i^2} F\left(\frac{(b-a)z_i+(b+a)}{2}\right)$$

Los puntos y pesos de integración de esta cuadratura se pueden determinar de forma explícita:

$$z_i = \cos\left(\frac{(2i+1)\pi}{2n+2}\right), \quad \omega_i = \frac{\pi}{n+1}\ , \quad i = 0, \ldots, n;$$

n	i	z_i	ω_i	$\omega_i \sqrt{1-z_i^2}$
0	0	0.00000000000000E+00	0.31415926535898E+01	0.31415926535898E+01
1	0	-0.70710678118655E+00	0.15707963267949E+01	0.11107207345396E+01
	1	0.70710678118655E+00	0.15707963267949E+01	0.11107207345396E+01
2	0	-0.86602540378444E+00	0.10471975511966E+01	0.52359877559830E+00
	1	0.00000000000000E+00	0.10471975511966E+01	0.10471975511966E+01
	2	0.86602540378444E+00	0.10471975511966E+01	0.52359877559830E+00
3	0	-0.92387953251129E+00	0.78539816339745E+00	0.30055886494217E+00
	1	-0.38268343236509E+00	0.78539816339745E+00	0.72561328803486E+00
	2	0.38268343236509E+00	0.78539816339745E+00	0.72561328803486E+00
	3	0.92387953251129E+00	0.78539816339745E+00	0.30055886494217E+00
4	0	-0.95105651629515E+00	0.62831853071796E+00	0.19416110387255E+00
	1	-0.58778525229247E+00	0.62831853071796E+00	0.50832036923153E+00
	2	0.00000000000000E+00	0.62831853071796E+00	0.62831853071796E+00
	3	0.58778525229247E+00	0.62831853071796E+00	0.50832036923153E+00
	4	0.95105651629515E+00	0.62831853071796E+00	0.19416110387255E+00
5	0	-0.96592582628907E+00	0.52359877559830E+00	0.13551733511720E+00
	1	-0.70710678118655E+00	0.52359877559830E+00	0.37024024484653E+00
	2	-0.25881904510252E+00	0.52359877559830E+00	0.50575757996373E+00
	3	0.25881904510252E+00	0.52359877559830E+00	0.50575757996373E+00
	4	0.70710678118655E+00	0.52359877559830E+00	0.37024024484653E+00
	5	0.96592582628907E+00	0.52359877559830E+00	0.13551733511720E+00
6	0	-0.97492791218182E+00	0.44879895051283E+00	0.99867161626728E-01
	1	-0.78183148246803E+00	0.44879895051283E+00	0.27982156872965E+00
	2	-0.43388373911756E+00	0.44879895051283E+00	0.40435388235934E+00
	3	0.00000000000000E+00	0.44879895051283E+00	0.44879895051283E+00
	4	0.43388373911756E+00	0.44879895051283E+00	0.40435388235934E+00
	5	0.78183148246803E+00	0.44879895051283E+00	0.27982156872965E+00
	6	0.97492791218182E+00	0.44879895051283E+00	0.99867161626728E-01
7	0	-0.98078528040323E+00	0.39269908169872E+00	0.76611790304042E-01
	1	-0.83146961230255E+00	0.39269908169872E+00	0.21817192032594E+00
	2	-0.55557023301960E+00	0.39269908169872E+00	0.32651735321160E+00
	3	-0.19509032201613E+00	0.39269908169872E+00	0.38515347895797E+00
	4	0.19509032201613E+00	0.39269908169872E+00	0.38515347895797E+00
	5	0.55557023301960E+00	0.39269908169872E+00	0.32651735321160E+00
	6	0.83146961230255E+00	0.39269908169873E+00	0.21817192032594E+00
	7	0.98078528040323E+00	0.39269908169872E+00	0.76611790304042E-01
9	0	-0.98768834059514E+00	0.31415926535898E+00	0.49145336613862E-01
	1	-0.89100652418837E+00	0.31415926535898E+00	0.14262532187813E+00
	2	-0.70710678118655E+00	0.31415926535898E+00	0.22214414690792E+00
	3	-0.45399049973955E+00	0.31415926535898E+00	0.27991795506908E+00
	4	-0.15643446504023E+00	0.31415926535898E+00	0.31029144348500E+00
	5	0.15643446504023E+00	0.31415926535898E+00	0.31029144348500E+00
	6	0.45399049973955E+00	0.31415926535898E+00	0.27991795506908E+00
	7	0.70710678118655E+00	0.31415926535898E+00	0.22214414690792E+00
	8	0.89100652418837E+00	0.31415926535898E+00	0.14262532187813E+00
	9	0.98768834059514E+00	0.31415926535898E+00	0.49145336613862E-01

$$R_n = \left(\frac{b-a}{2}\right) \frac{2\pi}{2^{2n+2}(2n+2)!}\, f^{(2n+2)}(\xi), \quad \xi \in [-1,+1]$$

Cuadratura de GAUSS-RADAU

$$\int_{-1}^{+1} f(z)dz \approx \sum_{i=0}^{i=n} \omega_i\, f(z_i) + R_n \qquad \int_a^b F(x)dx \approx \frac{b-a}{2} \sum_{i=0}^{i=n} \left[\omega_i\, F\left(\frac{(b-a)z_i + (b+a)}{2}\right)\right] + \frac{b-a}{2} R_n$$

n	i	z_i	ω_i
1	0	-0.10000000000000E+01	0.50000000000000E+00
	1	0.33333333333333E+00	0.15000000000000E+01
2	0	-0.10000000000000E+01	0.22222222222222E+00
	1	-0.28989794855664E+00	0.10249716523768E+01
	2	0.68989794855664E+00	0.75280612540093E+00
3	0	-0.10000000000000E+01	0.12500000000000E+00
	1	-0.57531892352169E+00	0.65768863996012E+00
	2	0.18106627111853E+00	0.77638693768634E+00
	3	0.82282408097459E+00	0.44092442235354E+00
4	0	-0.10000000000000E+01	0.80000000000000E-01
	1	-0.72048027131244E+00	0.44620780216714E+00
	2	-0.16718086473783E+00	0.62365304595148E+00
	3	0.44631397272375E+00	0.56271203029892E+00
	4	0.88579160777096E+00	0.28742712158245E+00
5	0	-0.10000000000000E+01	0.55555555555555E-01
	1	-0.80292982840235E+00	0.31964075322051E+00
	2	-0.39092854670727E+00	0.48538718846897E+00
	3	0.12405037950523E+00	0.52092678318957E+00
	4	0.60397316425278E+00	0.41690133431191E+00
	5	0.92038028589706E+00	0.20158838525348E+00
6	0	-0.10000000000000E+01	0.40816326530612E-01
	1	-0.85389134263948E+00	0.23922748922531E+00
	2	-0.53846772406011E+00	0.38094987364423E+00
	3	-0.11734303754310E+00	0.44710982901457E+00
	4	0.32603061943769E+00	0.42470377900596E+00
	5	0.70384280066303E+00	0.31820423146730E+00
	6	0.94136714568043E+00	0.14898847111202E+00
7	0	-0.10000000000000E+01	0.31250000000000E-01
	1	-0.88747487892616E+00	0.18535815480298E+00
	2	-0.63951861652622E+00	0.30413062064679E+00
	3	-0.29475056577366E+00	0.37651754538912E+00
	4	0.94307252661111E-01	0.39157216745249E+00
	5	0.46842035443082E+00	0.34701479563450E+00
	6	0.77064189367819E+00	0.24964790132987E+00
	7	0.95504122712257E+00	0.11450881474426E+00
8	0	-0.10000000000000E+01	0.24691358024691E-01
	1	-0.91073208942006E+00	0.14765401904632E+00
	2	-0.71126748591571E+00	0.24718937820459E+00
	3	-0.42635048571114E+00	0.31684377567044E+00
	4	-0.90373369606853E-01	0.34827300277297E+00
	5	0.25613567083346E+00	0.33769396697593E+00
	6	0.57138304120874E+00	0.28638669635723E+00
	7	0.81735278420041E+00	0.20055329802455E+00
	8	0.96444016970527E+00	0.90714504923283E-01

$$R_n = \frac{2^{2n+1}(n+1)(n!)^4}{((2n+1)!)^3}\, f^{(2n+1)}(\xi), \quad \xi \in [-1,1]$$

Cuadratura de GAUSS-LOBATTO

$$\int_{-1}^{+1} f(z)dz \approx \sum_{i=0}^{i=n} \omega_i\, f(z_i) + R_n \qquad \int_a^b F(x)dx \approx \frac{b-a}{2}\sum_{i=0}^{i=n}\left[\omega_i\, F\left(\frac{(b-a)z_i+(b+a)}{2}\right)\right] + \frac{b-a}{2}R_n$$

n	i	z_i	ω_i
2	0	-0.10000000000000E+01	0.33333333333333E+00
	1	0.00000000000000E+00	0.13333333333333E+01
	2	0.10000000000000E+01	0.33333333333333E+00
3	0	-0.10000000000000E+01	0.16666666666667E+00
	1	-0.44721359549996E+00	0.83333333333333E+00
	2	0.44721359549996E+00	0.83333333333333E+00
	3	0.10000000000000E+01	0.16666666666667E+00
4	0	-0.10000000000000E+01	0.10000000000000E+00
	1	-0.65465367070798E+00	0.54444444444444E+00
	2	0.00000000000000E+00	0.71111111111111E+00
	3	0.65465367070798E+00	0.54444444444444E+00
	4	0.10000000000000E+01	0.10000000000000E+00
5	0	-0.10000000000000E+01	0.66666666666666E-01
	1	-0.76505532392946E+00	0.37847495629785E+00
	2	-0.28523151648065E+00	0.55485837703549E+00
	3	0.28523151648065E+00	0.55485837703549E+00
	4	0.76505532392946E+00	0.37847495629785E+00
	5	0.10000000000000E+01	0.66666666666666E-01
6	0	-0.10000000000000E+01	0.47619047619048E-01
	1	-0.83022389627857E+00	0.27682604736157E+00
	2	-0.46884879347071E+00	0.43174538120986E+00
	3	0.00000000000000E+00	0.48761904761905E+00
	4	0.46884879347071E+00	0.43174538120986E+00
	5	0.83022389627857E+00	0.27682604736157E+00
	6	0.10000000000000E+01	0.47619047619048E-01
7	0	-0.10000000000000E+01	0.35714285714286E-01
	1	-0.87174014850961E+00	0.21070422714351E+00
	2	-0.59170018143314E+00	0.34112269248350E+00
	3	-0.20929921790248E+00	0.41245879465870E+00
	4	0.20929921790248E+00	0.41245879465870E+00
	5	0.59170018143314E+00	0.34112269248350E+00
	6	0.87174014850961E+00	0.21070422714351E+00
	7	0.10000000000000E+01	0.35714285714286E-01
8	0	-0.10000000000000E+01	0.27777777777778E-01
	1	-0.89975799541146E+00	0.16549536156081E+00
	2	-0.67718627951074E+00	0.27453871250016E+00
	3	-0.36311746382618E+00	0.34642851097305E+00
	4	0.00000000000000E+00	0.37151927437642E+00
	5	0.36311746382618E+00	0.34642851097305E+00
	6	0.67718627951074E+00	0.27453871250016E+00
	7	0.89975799541146E+00	0.16549536156081E+00
	8	0.10000000000000E+01	0.27777777777778E-01

$$R_n = \frac{-2^{2n+1}(n+1)(n!)^4}{n(2n+1)((2n)!)^3}\, f^{(2n)}(\xi), \quad \xi \in [-1,1]$$

Generación de PUNTOS y PESOS de Cuadraturas de GAUSS

Se emplea la subrutina DGQRUL de la Librería Matemática IMSL.
Los tipos de Cuadratura (NTYPE) que se pueden generar son:

```
1        LEGENDRE
2        TCHEBYSHEV de Primera Clase
3        TCHEBYSHEV de Segunda Clase
4        HERMITE
5        JACOBI
6        LAGUERRE
7        COSH
8        RADAU-LEGENDRE
9        LOBATTO-LEGENDRE
```

Compilación: `FOR CUADRATURAS`
Linkado: `LINK CUADRATURAS, IMSLIBG_STATIC/OPT, IMSLPSECT/OPT`

```
      program CUADRATURAS
      parameter (maxdim=30)
      implicit real*8(a-h,o-z),integer*4(i-n)
      dimension qw(maxdim),qx(maxdim),qxfix(2)
      external dgqrul
      open(11,file='puntos.dat',status='new')
      write(*,*)' Tipo de Cuadratura >> '
      read(*,*)ntype
    1 write(*,*)' Numero de Puntos de la Cuadratura >> '
      read(*,*)npuntos
      if(npuntos.gt.maxdim)goto1
      nfix=0
      itype=ntype
      if(ntype.eq.8)then
         itype=1
         nfix=1
         qxfix(1)=-1.0d+00
         if(npuntos.le.nfix)goto1
      elseif(ntype.eq.9)then
         itype=1
         nfix=2
         qxfix(1)=-1.0d+00
         qxfix(2)=+1.0d+00
         if(npuntos.le.nfix)goto1
      endif
      alfa=0.d+00
      beta=0.d+00
      call dgqrul(npuntos,itype,alfa,beta,nfix,qxfix,qx,qw)
      write(11,'(a,i3)')' Cuadratura tipo :', ntype
      write(11,'(a,i3)')' Numero de puntos :', npuntos
      do i=1,npuntos
         write(11,'(8x,i5,2x,e22.14,e22.14)')i-1,qx(i),qw(i)
      enddo
      close(11)
      end
```